Diffusive behavior from a quantum master equation

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We study a general class of translation invariant quantum Markov evolutions for a particle on $\mathbb{Z}^d$. The evolution consists of free flow, interrupted by scattering events. We assume spatial locality of the scattering events and exponentially fast relaxation of the momentum distribution. It is shown that the particle position diffuses in the long time limit. This generalizes standard results about central limit theorems for classical (non-quantum) Markov processes.


I. INTRODUCTION

A classical problem in the study of dynamical systems is understanding diffusive behavior for some properly rescaled variable. When starting from a Hamiltonian dynamics, that often proceeds in two steps. First, there is an identification of relevant space-time scales under which certain variables obey a reduced autonomous description. That specifies the limit starting from a microscopic dynamics and leading to a translation invariant master or Boltzmann-type equation, e.g., as the result of a weak coupling or a low density approximation.9,21,22 Already there some irreversible behavior may be exhibited. Additionally, a second step can further specify the irreversible properties of a more restricted set of degrees of freedom.

The present paper deals with the second step in a quantum setup, taking for granted a form of the master equation for the reduced description of a quantum particle hopping on the lattice. We imagine a translation invariant law of motion wherein the free Hamiltonian evolution is interrupted by scattering events from interactions with the environment. The effective or resulting description is that of a Markovian open system. In quantum mechanics, Lindblad equations take the place of Langevin or Fokker-Planck equations in classical probability theory describing a dynamical system under the influence of an idealized noisy environment (cf. Ref. 2). This Lindblad equation is a master equation for the evolution of the density matrix. The models we study in the present paper are translation invariant Markovian evolutions for a quantum particle on $\mathbb{Z}^d$. Its state is described via a density matrix $\rho_t$ for which, in position representation, the diagonal $\rho_t(x, x)$ gives the probability of the spatial location being $x$ at time $t$. The specific derivation of such master equations, very much like a linear Boltzmann equation, starting from the unitary evolution of a particle in contact with a reservoir is not the subject of the paper. Recently, some rigorous derivations of this type have been carried out successfully by several authors (see, e.g., Refs. 5–8). We do not discuss these derivations here, and we just restrict ourselves to the remark that not all of them lead to a quantum Markov process in the sense meant in this paper, namely as a Markovian equation for the density matrix of the particle, also called Lindblad equation. This comes about because the object that admits a scaling limit is often a Wigner function rather than a density matrix (for the works mentioned above, this is the case in Refs. 5, 7, and 8, only in Ref. 6 there is a limiting Markovian density matrix). A heuristic

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derivation of a translation invariant Lindblad equation has been advocated recently in Ref. 13 and see Ref. 24 for a review.

We show that under the right conditions, the solution of the quantum master equation behaves diffusively, exactly the same result as for its classical counterpart: the spatial probability density tends to a Gaussian, after rescaling the position as $x^2 \sim t$ and subtracting a possible systematic drift. We will also give some counterexamples to that result thereby making it less intuitive from a particular point of view. The questions and the applied techniques are, however, quite similar to what has been studied starting from classical Boltzmann-type equations, see, e.g., Refs. 3, 20, and 23. A quantum example, that is very related to ours is in Ref. 15. Also, the diffusive limit for another quantum master equation is studied in Ref. 4.

Intuitively, the dynamics we consider for the particle describes a free ballistic motion which gets interrupted by scattering events with a background fluid in which momentum is transferred to the particle. The background fluid is homogeneous and the interaction with the particle is translation invariant. The intuition for the particle-environment interaction as occurring through scattering events breaks down somewhat, since, in general, the environment also induces “spatial jumps” for the particle. This jumping is apparent in a contribution to the diffusion rate which is not driven by the ballistic motion.

Section II presents our model and gives a statement of the main result in Theorem 2.3. In Sec. III, we discuss some examples and a classical analogue. The rest of the paper contains the proof of the main theorem.

II. MODEL AND RESULTS

A. A translation covariant semigroup

For the open system dynamics which we consider, the state of the particle at a fixed time $t \geq 0$ is represented by a density matrix $\rho_t$ in $B_1(\ell^2(\mathbb{Z}^d))$, where $B_1(\ell^2(\mathbb{Z}^d))$ is the space of trace class operators over the Hilbert space $\ell^2(\mathbb{Z}^d)$. To be density matrices the $\rho_t$’s must also satisfy $\text{Tr}[\rho_t] = 1$ and have non-negative eigenvalues. The state $\rho_t$ evolves from an initial state $\rho$ as $\Phi_t(\rho) = \rho_t$, for a family of dynamical maps $\Phi_t : B_1(\ell^2(\mathbb{Z}^d)) \to B_1(\ell^2(\mathbb{Z}^d))$, $t \geq 0$ which are norm-continuous, form a semigroup $\Phi_t\Phi_s = \Phi_{t+s}$, preserve trace: $\text{Tr}\Phi_t(\rho) = \text{Tr}\rho$, and are completely positive, see, e.g., Ref. 2. By Ref. 19, a semigroup $\Phi_t$ with these properties must satisfy

$$\frac{d}{dt} \Phi_t(\rho) = L(\Phi_t(\rho)), \quad (2.1)$$

where $L$ can be written in the Lindblad form

$$L(\rho) = -i[H, \rho] + \Psi(\rho) - \frac{1}{2}\{\Psi^*(I), \rho\}, \quad \rho \in B_1(\ell^2(\mathbb{Z}^d)), \quad (2.2)$$

in which $H$ is a self-adjoint operator on $\ell^2(\mathbb{Z}^d)$, $\Psi$ is a completely positive map acting on $B_1(\ell^2(\mathbb{Z}^d))$, $\Psi^*$ is its adjoint, acting on $B(\ell^2(\mathbb{Z}^d))$, and $I$ is the identity operator on $\ell^2(\mathbb{Z}^d)$. Both $H$ and $\Psi$ are bounded and their forms have further restrictions discussed below when the $\Phi_t$’s have an additional spatial symmetry corresponding to an homogeneous environment.

To discuss the spatial symmetry that we assume and its consequences, we must define some operators associated with the position and momentum of the particle. On the Hilbert space $\ell^2(\mathbb{Z}^d)$, we define the translation operators $\tau_y$, $y \in \mathbb{Z}^d$, and the (vector-valued) position operator $X$ as

$$(\tau_y f)(x) = f(x + y), \quad (X_j f)(x) = x_j f(x) \quad \text{for} \quad f \in \ell^2(\mathbb{Z}^d), \quad x \in \mathbb{Z}^d.$$  

Here, and in what follows, the subscript $j = 1, \ldots, d$ refers to the components in $\mathbb{Z}^d$ or $\mathbb{T}^d$. We will often consider the space $\mathcal{H}$ in its dual representation, i.e., as $L^2(\mathbb{T}^d)$, where $\mathbb{T}^d$ is identified with $[-\pi, \pi]^d$. For $g \in L^2(\mathbb{T}^d)$, we define the vector of “momentum” operators $\Omega = (\Omega_j)$ as multiplication by $k \in \mathbb{T}^d$, i.e.,

$$\Omega_j g(k) = k_j g(k), \quad k \in [-\pi, \pi]^d.$$  

$(2.3)$
Although the $\Omega_j$’s are well-defined as bounded operators, they do not satisfy $[X_j, \Omega_j] = i$, nor does $\Omega$ generate the translations $\tau_x$. Trace class operators $\rho$ are Hilbert–Schmidt and thus have well-defined square integrable kernels $\rho(x_1, x_2) : \mathbb{Z}^d \to \mathbb{C}$ and $\rho(k_1, k_2) : \mathbb{T}^d \to \mathbb{C}$, which are related by the Fourier transform

$$\rho(k_1, k_2) := \frac{1}{(2\pi)^d} \sum_{x_1, x_2} e^{-i(k_1 x_1 - k_2 x_2)} \rho(x_1, x_2), \quad k_1, k_2 \in \mathbb{T}^d. \quad (2.4)$$

We refer to $\rho(x_1, x_2)$ and $\rho(k_1, k_2)$ as the position and momentum representation of the state $\rho$, respectively.

We demand that the semigroup $\Phi_t$ be translation covariant

$$\Phi_t(\tau_x \rho) = \tau_x \Phi_t(\rho), \quad x \in \mathbb{Z}^d.$$ 

By Ref. 11, this implies that one can choose $H = H(\Omega)$, a bounded function of the vector of momentum operators $\Omega$ and that $\Psi$ has what we refer to as the Holevo-form:

$$\Psi(\rho) = \int_{\mathbb{T}^d} d\nu(\theta) e^{i\theta X} \mathcal{M}_\theta(\rho) e^{-i\theta X}, \quad (2.5)$$

where $\nu$ is a positive finite measure on the $d$–dimensional torus $\mathbb{T}^d$, and the maps $\mathcal{M}_\theta$ are completely positive and act as multiplication in the momentum representation

$$\mathcal{M}_\theta(\rho)(k_1, k_2) = \rho(k_1, k_2)\rho(k_1, k_2), \quad (2.6)$$

for some functions $\mathcal{M}_\theta(k_1, k_2) : \mathbb{T}^d \to \mathbb{C}$. Note that the vector of position operators $X$ generates torus translations in the variable $k$, i.e., $e^{-i\theta X} \psi(k) = \psi(k + \theta)$, for $\psi \in L^2(\mathbb{T}^d)$ and thus, in momentum representation,

$$(e^{i\theta X} \rho e^{-i\theta X})(k_1, k_2) = \rho(k_1 - \theta, k_2 - \theta). \quad (2.7)$$

Boundedness of the functions $H(k)$ and $\int_{\mathbb{T}^d} d\nu \mathcal{M}_\theta(k, k)$ in $k \in \mathbb{T}^d$ is equivalent to the translation covariant semigroup being norm-continuous. Since the Hamiltonian $H(\Omega)$ is a function of the vector of momentum operators $\Omega$, the dynamics is driven by its momentum. Intuitively, the map $\mathcal{M}_\theta$ encodes the frequency and nature of the scattering which the particle passes through collisions with the reservoir that result in a momentum transfer $\theta$. As mentioned in Sec. II B, the maps $\mathcal{M}_\theta$ also generate spatial jumps for the particle which contribute to the diffusion constant (see the end of Sec. III A).

There is a crucial decomposition of the dynamics in the momentum representation as a consequence of the translation symmetry. It is useful to change variables in the momentum representation and to write

$$[\rho]_p(k) := \rho(k - \frac{p}{2}, k + \frac{p}{2}), \quad k, p \in \mathbb{T}^d, \quad (2.8)$$

where we will think of $[\rho]_p$ as fibers of the density matrix $\rho$, indexed by $p \in \mathbb{T}^d$. The Eq. (2.8) defines a map

$$[\cdot]_p : B_1(\ell^2(\mathbb{Z}^d)) \to L^1(\mathbb{T}^d).$$

We will study these maps with more care in Lemma 3.1. In particular, our conditions on the initial state $\rho$ will ensure that the function $p \mapsto [\rho]_p$ can be chosen in $C(\mathbb{T}^d, L^1(\mathbb{T}^d))$. Due to the translation symmetry, the dynamics gives rise to an autonomous evolution for each fiber $[\rho]_p$. This can be seen from (2.5)–(2.7) and the fact that both $\Psi^t(I)$ and $H$ are functions of the momentum operator and, thus, act as multiplication in momentum space (see also below under (3.6)). In particular, the momentum distribution for the particle, given by the diagonal $[\rho]_p(k) = \rho(k, k)$ in the momentum representation, undergoes a classical Markovian evolution.

The position distribution for the particle is found on the diagonal in the position representation, $\rho_0(x, x)$, which is itself not Markovian. The main and mathematical result of the paper is the identification of natural assumptions on the Hamiltonian $H$ and the maps $\mathcal{M}_\theta$ under which the
measure \( \mu_t \), defined by
\[
\mu_t(R) = \sum_{x \in \sqrt{t}R - tv} \rho_t(x, x),
\]
for a drift velocity \( v \in \mathbb{R}^d \) and for an arbitrary Borel set \( R \subset \mathbb{R}^d \), converges in distribution to a Gaussian law. Our mathematical assumptions on the kernels \( M_\theta(k_1, k_2) \) basically express a certain locality in the spatial jumps and a sufficiently smooth relaxation for the momentum distribution.

### B. Assumptions

There are basically two sets of assumptions, one having to do with the spatial locality of the dynamics and the other with the dissipativity. We formulate these conditions below, and we discuss them more closely in Sec. III after having stated the main result of the paper.

The locality can first be formulated in terms of the completely positive maps \( M_\theta \), the measure \( dv(\cdot) \) and the dispersion law \( H \). Note that there is a slight arbitrariness in choosing \( dv \) and \( M_\theta \) since only the measures \( dv(\cdot)M_\theta(k_1, k_2) \) enter in the definition of \( \Psi \).

**Assumption 2.1: [Locality]** We assume that the completely positive maps \( M_\theta \) are defined by the kernels \( M_\theta(k_1, k_2) \) as
\[
(M_\theta \rho)(k_1, k_2) = M_\theta(k_1, k_2) \rho(k_1, k_2),
\]
where the function \( M_\theta(k_1, k_2) \) is twice continuously differentiable in \( (k_1, k_2) \). Moreover, the matrix of derivatives \( D^2 M_\theta(k_1, k_2) \) is uniformly bounded in \( \theta, k_1, k_2 \in \mathbb{T}^d \), and the family of functions \( D^2 M_\theta, \theta \in \mathbb{T}^d \) is equicontinuous. The function \( H \) is assumed to be twice continuously differentiable. Finally, we assume that \( dv(\cdot) \) in (2.5) is a probability measure.

It is instructive to examine the map \( \Psi \) in the position representation. Let \( \tilde{M}_\theta : \mathbb{Z}^d \to \mathbb{C} \) be the double Fourier transform
\[
\tilde{M}_\theta(x_1, x_2) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} dk_1 dk_2 e^{-ik_1 x_1 + ik_2 x_2} M_\theta(k_1, k_2).
\]
The operation of \( \Psi \) in the position representation has the form
\[
\Psi(\rho)(x_1, x_2) = \int_{\mathbb{T}^d} dv(\theta) e^{i(x_1 - x_2)\theta} \sum_{y_1, y_2 \in \mathbb{Z}^d} \tilde{M}_\theta(x_1 - y_1, x_2 - y_2) \rho(y_1, y_2)
= \sum_{y_1, y_2 \in \mathbb{Z}^d} N(x_1, y_1, x_2, y_2) \rho(y_1, y_2),
\]
where the second equality determines the values of the kernel \( N : \mathbb{Z}^d \to \mathbb{C} \). It is apparent from the above form that the noise term \( \Psi \) generates some spatial jumping unless \( N(x_1, y_1, x_2, y_2) \) vanishes whenever \( x_1 \neq y_1 \) or \( y_2 \neq x_2 \). The translation covariance of \( \Psi \) is expressed through the equality
\[
N(x_1, y_1, x_2, y_2) = N(x_1 + z, y_1 + z, y_2 + z, x_2 + z), \quad \text{for all } z \in \mathbb{Z}^d.
\]
The conditions on the \( M_\theta \) in Assumption 2.1 can be replaced by the assumption
\[
\sup_{x_1, x_2 \in \mathbb{Z}^d} \sum_{y_1, y_2 \in \mathbb{Z}^d} ((x_1 - y_1)^2 + (x_2 - y_2)^2)|N(x_1, y_1, x_2, y_2)| < \infty
\]
to attain our results. This locality condition can be compared with asking for finite variance in the jumps in a random walk and it is supposed to exclude super diffusive behavior.

We now come to the dissipativity. To formulate that conveniently, we introduce the Markov generator \( A \) on \( L^1(\mathbb{T}^d) \)
\[
(AF)(k) = \int_{\mathbb{T}^d} dv(\theta) r(k, k - \theta) f(k - \theta) - \int_{\mathbb{T}^d} dv(\theta) r(k + \theta, k) f(k),
\]
\[
\int_{\mathbb{T}^d} dv(\theta) r(k, k - \theta) f(k - \theta) - \int_{\mathbb{T}^d} dv(\theta) r(k + \theta, k) f(k).
\]
\[
\int_{\mathbb{T}^d} dv(\theta) r(k, k - \theta) f(k - \theta) - \int_{\mathbb{T}^d} dv(\theta) r(k + \theta, k) f(k).
\]
with the transition rates $r(k, k')$ defined by,
\begin{equation}
    r(k + \theta, k) := M_\theta(k, k) \geq 0.
\end{equation}

The measure $d\nu(\theta)r(k + \theta, k)$ gives the probability density per unit time of jumping to the state $k + \theta$, conditioned on being in the state $k$. As before, the parameter $\theta$ plays the role of the momentum transfer.

Remember that $r(\cdot, \cdot)$ is a $C^1$-function by Assumption 2.1. Obviously, one has $\|A f\|_1 \leq c\|f\|_1$ for any $f \in L^1(\mathbb{T}^d)$ and $c := 2\|r(\cdot, \cdot)\|_{\infty}$, and hence $A$ is the bounded generator of a contractive (and positive) semigroup on $L^1(\mathbb{T}^d)$, see the connection with the Hille-Yosida theory in, e.g., Ref. 18. We now discuss dissipativity, which refers to ergodic properties.

**Assumption 2.2: [Dissipativity]** We assume

1. A has a simple eigenvalue 0 with eigenvector $\mathcal{P} \in L^1(\mathbb{T}^d)$, normalized such that $\int_{\mathbb{T}^d} dk \mathcal{P}(k) = 1$
2. The eigenvalue 0 is separated from the rest of the $L^1$-spectrum by a gap $b_A$.

$b_A := -\sup \text{Re} (\text{spec}(A) \setminus \{0\}) > 0$

The above assumptions guarantee that the semigroup generated by $A$, i.e., $e^{tA}$, relaxes exponentially fast to the stationary distribution $\mathcal{P}$. For future use, we let $Y_t$ stand for the Markov process on $\mathbb{T}^d$ generated by $A$ and started from $\mathcal{P}$. Using standard techniques for Markov processes on compact spaces, constructive conditions are available for guaranteeing the Assumption 2.2 in terms of $r(k, k')$ and $v$. While the above assumptions are natural ergodicity or gap-assumptions, in fact for our result we need less. In particular, the exponential relaxation is not strictly necessary. We will however not describe that.

The following standard construction will be useful in the statement of one of our results. Consider the scalar product,
\begin{equation}
   \langle f, g \rangle_{\mathcal{P}} := \int dk \mathcal{P}(k) f(k) \overline{g(k)}, \quad f, g \in C(\mathbb{T}^d),
\end{equation}
and let $\mathcal{H}_{\mathcal{P}}$ stand for the Hilbert space which is the completion of $C(\mathbb{T}^d)$ with respect to the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{P}}$. Define the quadratic form $A_{\mathcal{P}}$ by
\begin{equation}
   \langle f, A_{\mathcal{P}} g \rangle_{\mathcal{P}} := \int dk \mathcal{P}(k) |A^* f(k)| \overline{g(k)}, \quad f, g \in C(\mathbb{T}^d),
\end{equation}
where $A^*$ is the adjoint of $A$ acting on $L^\infty(\mathbb{T}^d)$.

The operator $A$ appears naturally in a perturbation setup around the zero fiber. In fact, the evolution on the zero fiber is the Markov process generated by $A$.

We finally ask some properties that appear directly linked with the notion of diffusion. We have of course already that the particle must be sufficiently localized since it is described by a density matrix $\rho \in B_1(\ell^2(\mathbb{Z}^d))$, but we also ask that the first two moments are well-defined in the following way:
\begin{equation}
   X_j \rho, \quad X_i \rho X_j, \quad X_i X_j \rho \quad \text{are in } B_1(\ell^2(\mathbb{Z}^d)), \quad \text{for } i, j = 1, \ldots, d.
\end{equation}
Products of bounded and unbounded operators such as in (2.14) are to be understood as kernels of sesquilinear forms with the densely defined domains. For example, $X_j \rho \in B(\ell^2(\mathbb{Z}^d))$ (which is implied since $B_1 \subset B$) means that $b(f, g) := \langle X_j f, \rho g \rangle$ satisfies $|b(f, g)| < C \|f\|_2 \|g\|_2$ for $(f, g) \in \text{Dom}(X_j) \times \ell^2(\mathbb{Z}^d)$ and some $C < \infty$, and thus the quadratic form $b(\cdot, \cdot)$ is extendable to $\ell^2(\mathbb{Z}^d) \otimes \ell^2(\mathbb{Z}^d)$. In particular, note that by the boundedness of the form $b(\cdot, \cdot)$ and by the definition of the domain of the self-adjoint operator $X_j$, we have that
\begin{equation}
   \rho g \in \text{Dom}(X_j), \quad \text{for any } g \in \ell^2(\mathbb{Z}^d),
\end{equation}
so that the product $X_j \rho$ makes sense.
We write, in general, \( \rho_t \) for the solution of the Eq. (2.1) with initial condition \( \rho_0 \in \mathcal{B}_1 \). We will show that there is a \( v \in \mathbb{R}^d \) such that

\[
  v = \lim_{t \to \infty} \frac{1}{t} \sum_{x \in \mathbb{Z}^d} x \rho_t(x, x)
\]

(2.16)

That can be strengthened to a weak law of large numbers. One can obviously force \( v = 0 \) by requiring some additional symmetries. Getting our results, however, does not depend on these extra requirements. In particular, equilibrium conditions such as detailed balance are mostly irrelevant for the diffusive behavior around the drift, except when one wants, for example, to relate the diffusion constant to the mobility in linear response theory.

**C. Result**

We define \( T^{(1)} \) and \( T^{(2)} \) as, respectively, a vector and \( d \times d \) matrix of operators on \( L^1(\mathbb{T}^d) \), by

\[
  (T^{(1)} f)(k) = -i(\nabla H)(k) f(k) + \int_{\mathbb{T}^d} d\nu(\theta) m^{(1)}_\theta(k - \theta) f(k - \theta),
\]

(2.17)

where \( m^{(1)}_\theta(k) = -i \text{Im}(\nabla_1 M_\theta(k, k)) \) (and \( \nabla_1 \) and \( \nabla_2 \) are the gradients with respect to the first and second variables of \( M_\theta(k_1, k_2) \)), and

\[
  (T^{(2)} f)(k) = \int_{\mathbb{T}^d} d\nu(\theta) \left( (m^{(3)}_\theta(k - \theta) f(k - \theta) - m^{(3)}_\theta(k) f(k)) \right),
\]

(2.18)

where

\[
  (m^{(2)}_\theta(k))_{(i, j)} = \frac{1}{4} \left[ \{\nabla_1 \nabla_2 M_\theta(k, k)\}_{(i, j)} + \{\nabla_1 \nabla_2 M_\theta(k, k)\}_{(j, i)} \right],
\]

and

\[
  (m^{(3)}_\theta(k))_{(i, j)} = \frac{1}{2} \left[ \{\nabla^2_1 \nabla_2 M_\theta(k, k)\}_{(i, j)} + \{\nabla^2_1 \nabla_2 M_\theta(k, k)\}_{(j, i)} \right].
\]

We let \( P_0 \) be the spectral projection corresponding to the 0 eigenvalue of \( A \), the Markov generator defined in (2.11), and take \( S \) the reduced resolvent of \( A \) at the eigenvalue 0, i.e., the solution of

\[
  S(0 - A) = (0 - A)S = 1 - P_0, \quad SP_0 = P_0S = 0.
\]

Finally, recall that \( \mathcal{P} \) is the eigenvector corresponding to the eigenvalue 0. Hence \( \mathcal{P}(k) dk \) is the stationary probability measure on the torus. In general, the projection \( P_0 \) is non-orthogonal and has the form \( P_0 = |\mathcal{P}|(1_{\mathbb{T}^d}) \), where \( 1_{\mathbb{T}^d} \) is the constant function on \( \mathbb{T}^d \) with value 1. We use the notation \( \langle g, f \rangle := \int_{\mathbb{T}^d} dk \mathcal{P}(k) f(k) \) for the pairing between \( f \in L^1(\mathbb{T}^d) \) and \( g \in L^\infty(\mathbb{T}^d) \). To the time-evolved density matrix \( \rho_t \), we associate the measures \( \mu_t \) on \( \mathbb{R}^d \), defined by

\[
  \mu_t(R) := \text{Tr}[1_{\sqrt{t} R - v t}(X) \rho_t], \quad \text{for a Borel set } R \subset \mathbb{R}^d,
\]

(2.19)

where \( 1_{\sqrt{t} R - v t}(X) \) is the spectral projection of the vector of position operators \( X \) on the set \( \sqrt{t} R - v t \subset \mathbb{R}^d \).

**Theorem 2.3:** Take Assumptions 2.1 and 2.2 and let the initial density matrix \( \rho_0 \in \mathcal{B}_1(\ell^2(\mathbb{Z}^d)) \) satisfy (2.14). The limiting velocity exists and equals

\[
  v := \lim_{t \to \infty} \frac{1}{t} \sum_{x \in \mathbb{Z}^d} x \rho_t(x, x) = -i(1_{\mathbb{T}^d}, T^{(1)} \mathcal{P}).
\]

(2.20)

The measure \( \mu_t \), defined as in (2.19) with \( v \) as in (2.20), converges, as \( t \to \infty \), in distribution to a Gaussian with covariance matrix \( \sigma = \beta + \beta^\dagger \), where \( \beta^\dagger \) is the transpose of the matrix \( \beta \), given by

\[
  \beta := -(1_{\mathbb{T}^d}, (T^{(2)} - T^{(1)} ST^{(1)} \mathcal{P}),
\]

(2.21)

where the operators \( T^2 \) and \( T^1 \) as defined above.
The truncated second moments converge to the covariance matrix $\sigma$, i.e.,
\[
\lim_{t,\nu \to \infty} \sum_{x \in \mathbb{Z}^d} \rho_t(x, x)(x_i - \nu t)(x_j - \nu t) = \sigma(i, j). \tag{2.22}
\]

Equation (2.22) tells us that the covariance matrix $\sigma$ can be interpreted as the diffusion tensor. In the case that $T^{(1)} = -i \nabla H(\Omega)$ and $T^{(2)} = 0$, it reduces to another matrix that below we call $\alpha$. In the proposition below, the matrices $\sigma$ and $\alpha$ are “non-negative” in the sense of real valued vectors: $\forall (v(\in \mathbb{R}^d), (v, \sigma v), (v, \alpha v) \geq 0$.

**Proposition 2.4:** Consider the covariance matrix $\sigma$ as above.

1. $\sigma$ is non-negative.
2. Define the (vector) function
   \[
   \zeta := \nabla H - \langle \nabla H, \mathcal{P} \rangle \tag{2.23}
   \]
on $\mathbb{T}^d$. The real-valued $d \times d$–matrix $\alpha$ with entries
   \[
   \alpha(i, j) = \frac{1}{2} \int_0^\infty dt \mathbb{E}_\mathcal{P}[\langle \zeta(Y_t) \rangle_i \langle \zeta(Y_t) \rangle_j + \langle \zeta(Y_t) \rangle_i \langle \zeta(Y_t) \rangle_j]. \tag{2.24}
   \]
is non-negative. (As in Sec. II B, $Y_t$ is the stationary Markov process on $\mathbb{T}^d$ generated by $A$ and started from $\mathcal{P}(k)dk$, cf. (2.11)).
3. Assume that for all $w \in \mathbb{R}^d$, the function $k \mapsto \langle w, \zeta(k) \rangle$ does not vanish identically (we write $(\cdot, \cdot)$ for the scalar product on $\mathbb{R}^d$). That is, all components of the velocity $\nabla H$ can fluctuate. Assume in addition that $A_{\mathcal{P}}$, defined as a quadratic form in (2.13), extends to a bounded and sectorial operator on $\mathcal{H}_\mathcal{P}$. Sectoriality means that there is $0 < \gamma < \infty$ such that
   \[
   |\langle f, \text{Im}(A_{\mathcal{P}})f \rangle_{\mathcal{P}}| \leq -\gamma \langle f, \text{Re}(A_{\mathcal{P}})f \rangle_{\mathcal{P}}, \quad \text{for all } f \in \mathcal{H}_\mathcal{P}. \tag{2.25}
   \]
   Then the matrix $\alpha$ is strictly positive (i.e., it has strictly positive eigenvalues).

III. DISCUSSION

A. Properties of the drift and diffusion constant

We can erase drift terms and simplify the diffusion matrix $\sigma$ by imposing further symmetries. Define the linear and antilinear maps $U$ and $V$ acting on $f \in \ell^2(\mathbb{Z}^d)$ as
\[
(Uf)(x) = f(-x) \quad \text{and} \quad (Vf)(x) = \overline{f}(x).
\]
If we assume the space-inversion symmetry
\[
U \Phi_t(\rho)U^{-1} = \Phi_t(U\rho U^{-1}), \quad \rho \in \mathcal{B}_1(\ell^2(\mathbb{Z}^d)) \tag{3.1}
\]
(or on the generator $L$ of the semigroup $\Phi_t$), we have no drift, i.e.,
\[
0 = v = \lim_{t,\nu \to \infty} \frac{1}{t} \sum_x x\rho_t(x, x). \tag{3.2}
\]
If we assume the symmetry (for the noise term)
\[
W \Psi(\rho)W^{-1} = \Psi(W\rho W^{-1}), \quad W = UV, \tag{3.3}
\]
then $M_0(\cdot, \cdot)$ is real, consequently the function $m_0^{(1)}$ in (2.17) is zero and $T^{(1)} = -i \nabla H$. This means that the drift is given by $v = (\nabla H, \mathcal{P})$ and the diffusion matrix $\sigma$ simplifies to the form
\[
\sigma(i, j) = \int_{\mathbb{T}^d} dk \int_{\mathbb{T}^d} dv(\theta) \mathbb{P}(k)(([\nabla_1 \nabla_2 M_0]_{i, j})(k, k) + [\nabla_1 \nabla_2 M_0]_{j, i})(k, k))
+ \int_0^\infty dt \mathbb{E}_\mathcal{P}[\langle \zeta(Y_t) \rangle_i \langle \zeta(Y_t) \rangle_j + \langle \zeta(Y_t) \rangle_j \langle \zeta(Y_t) \rangle_i]. \tag{3.4}
\]
where the rightmost expression is defined as in (2.24). Notice that the vanishing of \( m^{(3)}_0 \) from these expressions is a general feature that does not depend on (3.3). The second term in (3.4) is the diffusion constant of a classical Boltzmann equation with momentum evolution given by the Markov process \( Y_t \). The form makes the contribution of spatial jumps to the diffusion matrix \( \sigma \) clear. The first term on the RHS of (3.4) depends on the noise but not on the dispersion relation \( \nabla H \). This part of the diffusion must arise from the noise-induced spatial jumps related to the position representation (2.10). A special case, where this term disappears, is discussed in Sec. III C.

We sketch why this first term is a non-negative matrix. Since \( M_0 \) is a completely positive map on \( B_1 (\ell^2 (Z^d)) \), for each \( \theta \), it follows that \( (v, \sqrt{1} \nabla_2 M_k v) \) is the integral kernel of a positive operator on \( L^2 (\mathbb{T}^d) \). This follows since \( M_0 (\psi (f) f) \) is a positive bounded operator for any \( f \in L^2 (\mathbb{T}^d) \) and thus for any \( g \in L^2 (\mathbb{T}^d) \)

\[
0 \leq \langle g, M_0 (\psi (f) f) \rangle g = \int_{\mathbb{T}^d} M_0 (k_1, k_2) h(k_1) h(k_2) \quad h(k) = \tilde{f}(k) g(k). \tag{3.5}
\]

Through choice of \( f, g \), the functions \( h(k) \) can be formed into arbitrary elements in \( L^1 (\mathbb{T}^d) \). Choose \( h = (v \cdot \nabla h_0) (k) \) for any once continuously differentiable function \( h_0 (k) \). Using integration by parts twice, we have

\[
0 \leq \int_{\mathbb{T}^d} dk \ h_0 (k_1) h_0 (k_2) (v, [\sqrt{1} \nabla_2 M_k] v) (k_1, k_2).
\]

The positivity can be extended to all \( h_0 \in L^2 (\mathbb{T}^d) \), since \( M_0 (k_1, k_2) \) is twice continuously differentiable and thus \( (v, [\sqrt{1} \nabla_2 M_k] v) (k_1, k_2) \) determines a Hilbert-Schmidt operator. The diagonal entries in the integral kernel for a positive operator are non-negative, so

\[
(v, \int_{\mathbb{T}^d} dk \ d v(\theta) \mathcal{P}(k) [\sqrt{1} \nabla_2 M_k] (k, k) v)
\]

is thus non-negative as claimed. Symmetrizing \( [\sqrt{1} \nabla_2 M_k] (k, k) \) in the above expression makes no difference in the evaluation of expressions \( (v, \sigma v) \) for \( v \in \mathbb{R}^d \) but ensures that \( \sigma \) is a symmetric and real-valued diffusion matrix.

**B. Idea of proof: Perturbation theory**

The main feature of our translation invariant models is that the evolution generated by (2.1) can be decomposed along the fibers (2.8), i.e., one can write \( [L \rho]_p = L_p [\rho]_p \) for some operators \( L_p \) and the fibers of the density matrix, \( [\rho]_p \), obey the differential equation

\[
\frac{d}{dt} [\rho]_p = L_p [\rho]_p. \tag{3.6}
\]

The expression for \( L_p \) can be determined as a quadratic form through a trace formula: for \( F \in L^\infty (\mathbb{T}^d) \),

\[
\text{Tr} [L^* (e^{it^2 \phi X} F (\Omega) e^{-it^2 \phi X}) \rho] = \langle F, L_p [\rho]_p \rangle = \int_{\mathbb{T}^d} dk \ F (k) L_p [\rho]_p (k). \tag{3.7}
\]

A simple computation shows that \( L_p = -i \hbar_p + \varphi_p \) with

\[
(h_p f)(k) := (H(k - \frac{\hbar}{2}) - H(k + \frac{\hbar}{2})) f(k) \tag{3.8}
\]

\[
(\varphi_p f)(k) := \int_{\mathbb{T}^d} d v(\theta) r_p (k - \theta, k) f(k - \theta) - \bar{r}_p (k, k + \theta) f(k), \tag{3.9}
\]

where \( r_p (k, k') = M_{L -k} (k - \frac{\hbar}{2}, k + \frac{\hbar}{2}) \). Note that, \( r_{p=0} (k, k') \) was simply called \( r (k, k') \) in (2.12). By the inequality (3.5) for a sequence of \( h_n = h_n^\prime \) approaching a \( \delta \)-function, it follows that the values \( r (k, k') \) are non-negative.

Since only the fibers around \( p = 0 \) determine the diffusive behavior, this suggests using a perturbation argument to capture the essential dynamical properties of those fibers. Under our
assumptions on $A = L_0$, it has a gap between the zero eigenvalue corresponding to the stationary distribution $\mathcal{P}$ and the rest of the spectrum which has strictly negative real part. Hence, sufficiently small bounded perturbations of $L_0$ keep the gap open.

The following proposition gives a condition on $\rho$ such that the function $p \mapsto [\rho]_p$ is twice differentiable. This assures the existence of the first two moments.

**Proposition 3.1:** Assume that a density matrix $\rho \in B_1(\ell^2(\mathbb{Z}^d))$ satisfies

$$X_j \rho, \quad X_j \rho X_j, \quad X_i X_j \rho \quad \text{are in} \quad B_1(\ell^2(\mathbb{Z}^d)) \quad \text{for} \quad i, j = 1, \ldots, d. \quad (3.10)$$

(as in (2.14)), then the function $\mathbb{T}^d \to L^1(\mathbb{T}^d) : p \mapsto [\rho]_p$ can be chosen to be twice continuously differentiable.

**C. Examples**

We mention a certain subclass of models satisfying the conditions for (2.3) and another class of models that do not satisfy our conditions and that possibly show a non-diffusive (e.g., super-diffusive) behavior.

We consider first the case when the completely positive maps $M_\theta$ in (2.5) act identically $M_\theta \rho = \rho$, such that $\Psi$ becomes

$$\Psi(\rho) = \int_{\mathbb{R}^d} dv(\theta) e^{i\theta X} \rho e^{-i\theta X}. \quad (3.11)$$

The map $\Psi$ operates multiplicatively in the position representation

$$\Psi(\rho)(x, y) = \varphi(x - y) \rho(x, y),$$

where $\varphi$ is the Fourier transform of the measure $v$. A noise of the type (3.11) has appeared as a tight-binding approximation for the modeling of a low energy atom in a periodic potential. Also, it is similar to the noise term for Gallis-Flemming dynamics for a quantum particle in $\mathbb{R}^d$ that has been frequently discussed in the decoherence literature. The derivation in Ref. 10 starts from a scattering framework and considers a limiting regime where the mass of the particle is much larger than the mass of particles in a background gas. The jump rates $d\nu(k - k') \rho(k, k') = d\nu(k - k')$ from the zero-fiber process (2.11) only depend on the difference between the starting momentum $k'$ and landing momentum $k$. The dynamics can thus be described as frictionless, and the stationary density $\mathcal{P}$ of the zero-fiber Markov process is the uniform distribution. Since $\mathcal{V}_1 \mathcal{V}_2 M_{\theta}(k_1, k_2) = 0$, the diffusion constant $\sigma$ takes the familiar form of the second term in (3.4). That is, the diffusion matrix reduces to $\alpha$, as defined in Proposition 2.4.

Thinking about adding spatial jumps we arrive at models where the noise $\Psi$ resembles a simple symmetric random walk. Yet, that easily breaks Assumption 2.2 (Statement 2), and the model can become super-diffusive when the kinetic term is included. As an illustration, we consider a one-dimensional model. Let $\Psi$ have the form

$$\Psi(\rho)(x_1, x_2) = \sum_{y_1, y_2 \in \mathbb{Z}} N(x_1, y_1, y_2, x_2) \rho(y_1, y_2),$$

where $N(x_1, y_1, y_2, x_2)$ is determined by a function $r(x)$, $x \in \mathbb{Z}^d$, with a positive Fourier transform through the equation

$$N(x_1, y_1, y_2, x_2) = r(x_1 - x_2) \chi[|x_1 - y_1| = 1] \chi[|x_2 - y_2| = 1],$$

where $\chi[\cdot]$ is the indicator (1 or 0). In that case, the measure $d\nu(\cdot)$ can be taken to be Lebesgue measure and the functions $M_{\theta}(k_1, k_2)$ are determined by

$$M_{\theta}(k_1, k_2) = 4 \hat{\varphi}(\theta) \cos(k_1) \cos(k_2)$$

$$\Psi(\rho)(k_1, k_2) = 4 \int_T d\theta \hat{\varphi}(\theta) \cos(k_1 - \theta) \cos(k_2 - \theta) \rho(k_1 - \theta, k_2 - \theta). \quad (3.12)$$
The smoothness of \( \hat{r}(\theta) = \frac{1}{2\pi} \sum_n e^{in\theta} r(n) \) depends on the decay of \( c \). Notice that the Markov process of the zero fiber is generated by

\[
(Af)(k) = 4 \int_T d\theta \hat{r}(\theta) \left( \cos(k - \theta)^2 f(k - \theta) - \cos(k)^2 f(k) \right). \tag{3.13}
\]

That is describing a Markov process on the torus \( T \) where an escape from the point \( k \) occurs with rate \( 4 \cos^2(k) \left( \int_T d\theta \hat{r}(\theta) \right) \) and the jump size \( \theta \) is independent of \( k \) and has probability density

\[
\frac{\hat{r}(\theta)}{\int_T d\theta \hat{r}(\theta)}.
\]

Assumption 2.2 is now violated in that, for \( k = \pm \frac{\pi}{2} \),

\[
\int_T d\theta M_\theta(k, k) = 4(\int_T d\theta \hat{r}(\theta)) \cos^2(k) = 0.
\]

This implies that there are degenerate stationary distributions of the form

\[
\mathcal{P}_\lambda(k) = \lambda \delta(\frac{\pi}{2} - k) + (1 - \lambda) \delta\left(-\frac{\pi}{2} - k\right), \tag{3.14}
\]

for \( 0 \leq \lambda \leq 1 \). The process is thus slow to leave the regions around \( k = \pm \frac{\pi}{2} \). We conjecture that for an arbitrary smooth probability distribution \( \mathcal{V} \) on \( T \), \( e^{t\mathcal{H}} \mathcal{V} \) will converge in distribution to \( \mathcal{P}_{1/2} \), i.e., to (3.14) for \( \lambda = \frac{1}{2} \).

When the kinetic term \( H \) is zero, we observe that still the diffusion constant as such keeps making mathematical sense as

\[
\sigma = 2 \int_T dk \mathcal{P}_\lambda(k) (\partial_1 \partial_2 M)(k, k) = 8 \int_T d\theta \hat{r}(\theta).
\]

When the kinetic term \( H \) is non-zero and \( H'(\pm \frac{\pi}{2}) \neq 0 \), we expect that the model exhibits a behavior closer to being ballistic with a width in position that grows on the order of \( t \) rather than \( r^\frac{1}{2} \). The basic idea is that when the particle attains a momentum close to \( k = \pm \frac{\pi}{2} \), then it tends to move ballistically without interruption with that momentum for long intervals of time. A classical analogue of this process was studied in Sec. 4 of Ref. 14 for the case such that \( H'(k \pm \frac{\pi}{2}) \approx \pm c|k \pm \frac{\pi}{2}|' \), when \( |k \pm \frac{\pi}{2}| \ll 1 \) for some \( c > 0 \) and \( \gamma \geq 1 \). They consider a linear functional \( Y_t = \int_0^t dr V(K_r) \), for a dispersion relation \( V : T \rightarrow \mathbb{R} \) (e.g., \( V = H' \)) and Markov process \( K_r \) whose densities obey the master Eq. (3.13) (and for a more general class of jump rates). They show that \( N^{-\frac{1}{2}} Y_{Nt} \) converges in law to a limiting process as \( N \rightarrow \infty \) for \( \beta = \frac{1}{2} \) when \( \gamma > 1 \) and \( \beta = 1 \) when \( \gamma = 1 \). These correspond to situations where the velocities are small for the regions of momenta around \( \pm \frac{\pi}{2} \), where the escape rates are small (and thus the occupation times are large). The particle typically spends most of the time interval \( r \in [0, Nt] \) with momenta \( K_r \approx \pm \frac{\pi}{2} \) corresponding to small velocities. When \( 0 \leq \gamma < 1 \), then we conjecture that \( N^{-1+\frac{1}{2}} Y_{Nt} \) converges in law to a nontrivial limit. This would suggest that the quantum model discussed above behaves ballistically when \( H'(\pm \frac{\pi}{2}) \neq 0 \), corresponding to the case \( \gamma = 0 \).

D. A classical analogue

There is a sense in which our present quantum problem differs little from an analogous classical problem that starts from a linear Boltzmann equation. We explain that analogy here. Among other things it relates our results to the recent interest and work on diffusive behavior in systems of coupled oscillators where energy transport can be understood as a wave scattered by anharmonicities.\(^{20}\)
Consider a stochastic dynamics with state space $S = \mathbb{Z}^d \times \mathbb{T}^d$ such that the probability density $P_t(x, k) = \Gamma_t(P)(x, k)$ evolves as

$$
\frac{d}{dt} P_t(x, k) = -\sum_{j=1}^d \left[ (\nabla H)_j(k) \right] \left( P_t(x, k) - P_t(x - s_j(k)e_j, k) \right) + \int_{(x', k') \in S} \left( T(x, k; x', k') P_t(x', k') - T(x', k; x, k) P_t(x, k) \right),
$$

(3.15)

for an initial $P_0(x, k) = P(x, k)$, where the $e_j$ are the standard basis vectors of $\mathbb{Z}^d$, $H : \mathbb{T}^d \to \mathbb{R}$ is differentiable, $s_j(k)$ is the sign of $(\nabla H)_j(k)$, and $T(x, k; x', k') \geq 0$ is a transition matrix describing the rates of Poisson timed jumps from $(x', k')$ to $(x, k)$ and the symbol $\int_{(x', k') \in S}$ stands for $\int_{\mathbb{Z}^d} dk' \sum_{k \in \mathbb{Z}^d}$. The first term on the right generates spatial steps with Poisson rate $[(\nabla H)_j(k)]$ in the direction $s_j(k)e_j$. That term could be formally absorbed into the second term on the right side of (3.15), although it is analogous to the kinetic term for the quantum case in its mathematical treatment. If the particle were living on $\mathbb{R}^d$ rather than $\mathbb{Z}^d$, then the first term would have the less awkward form $-(\nabla H)(k) \cdot \nabla_x P_t(x, k)$ for a differentiable $P_t(x, k) \in L^1(\mathbb{R}^{2d})$, which describes a deterministic kinetic motion. The connection can be made through a formal limit

$$(\nabla H)(k) \cdot \nabla_x P_t(x, k) = \lim_{\epsilon \to 0} -\frac{1}{\epsilon} \sum_{j=1}^d \left[ (\nabla H)_j(k) \right] \left( P_t(x, k) - P_t(x - \epsilon s_j(k)e_j, k) \right), \quad x, k \in \mathbb{R}^d,
$$

which is essentially a law of large numbers from the stochastic point of view, where many small random jumps (of size $\epsilon \ll 1$) combine to form a deterministic quantity.

When $T$ satisfies $T(x + z, k; x' + z, k') = T(x, k; x', k')$, then the master Eq. (3.15) describes a formally translation invariant evolution. From a classical physics perspective the evolution (3.15) is still somewhat strange as it involves both jumps in position ($x$) and in momentum ($k$) for the single particle’s state evolution. Still one can ask the classical questions about its diffusive behavior. We must then show that the centered position $(x - vt)/\sqrt{t}$ converges in distribution to a Gaussian with non-degenerate covariance matrix. We show here how this can proceed along the very same lines as for the quantum case.

By translation invariance the dynamics does not “feel” the location of the particle and it follows that the “momentum” dynamics is Markovian, or the marginal probability, $\hat{P}_t(k) = \sum_{x \in \mathbb{Z}^d} P_t(x, k)$, for the $k$ variable obeys an autonomous first order equation:

$$
\frac{d}{dt} \hat{P}_t(k) = \int_{\mathbb{T}^d} dk' \left( \tilde{T}(k, k') \hat{P}_t(k') - \tilde{T}(k', k) \hat{P}_t(k) \right),
$$

(3.16)

where $\tilde{T}(k, k') = \sum_{z \in \mathbb{Z}^d} T(x + z, k; x, k')$. This is analogous to the zero fiber of our decomposition for the quantum dynamics. Defining $[\Gamma_t]_p(k) = \sum_{x \in \mathbb{Z}^d} e^{ipx} P_t(x, k)$, the dynamics (3.15) operates as $[\Gamma_t]_p[P]_p = [\Gamma_t(P)]_p$ for some semigroup $[\Gamma_t]_p : L^1(\mathbb{T}^d) \to L^1(\mathbb{T}^d)$ whose generator is written below.

The closest thing to a joint probability density for position and momentum in the quantum case is the Wigner function. Here, we go in the opposite direction and we define a “classical density matrix” from a joint distribution function $P(x, k)$. That is, for a momentum representation kernel, we formally define:

$$
\rho(k_1, k_2) = \sum_{x \in \mathbb{Z}^d} e^{-i(x-k_1-k_2)} P(x, \frac{k_1 + k_2}{2}).
$$

(3.17)

In this case, the positivity of $\rho$ is lost since it is not, in general, the kernel of an operator with positive spectrum. The invariant fibers again correspond to $p = k_1 - k_2$ in the momentum representation. The dynamics can thus be written in its fibers with $\frac{d}{dt}[\Gamma_t]_p$ operating as

$$
\frac{d}{dt} [P_t]_p(k) = -i[H]_p(k)[P_t]_p(k) + \int_{\mathbb{T}^d} dk' \left( \tilde{T}_p(k, k') [P_t]_p(k') - \tilde{T}_0(k', k) [P_t]_p(k) \right),
$$

(3.18)
where $h_p(k) = -i \sum_{j=1}^{d} (1 - e^{i\gamma(k,j)})(\nabla H_j)(k)$ and $\tilde{T}_p(k, k') = \sum_{z \in \mathbb{Z}^d} e^{ipz}T(x + z, k; x, k')$. One notices similarities with the fiber decomposition and with (3.8).

IV. PROOFS

We need a technical lemma to deal conveniently with trace class operators.

**Lemma 4.1:** Let $C \in \mathcal{B}_1(\mathcal{H})$, for some Hilbert space $\mathcal{H}$, and let $X$ be a self-adjoint operator. If $XC \in \mathcal{B}_1$, then

$$
\lim_{\gamma \to 0} \frac{1}{i\gamma}(e^{i\gamma X} - I)C = XC,
$$

where the convergence is meant in the sense of $\mathcal{B}_1(\mathcal{H})$.

**Proof:** Let

$$
C = \sum_{n \in \mathbb{N}} \lambda_n |f_n\rangle \langle g_n|, \quad f_n, g_n \in \mathcal{H}, \quad \sum_n |\lambda_n| < \infty,
$$

be the singular value decomposition of $C \in \mathcal{B}_1$. Both families $f_n$ and $g_n$ are an orthonormal set. By the comment (2.15) and $Cg_n = \lambda_n f_n$, it follows that when $\lambda_n \neq 0$, then $f_n \in \text{Dom}X$.

If there are only finitely many terms in the singular value decomposition (4.2), then the convergence (4.1) is guaranteed by an application of Stone’s theorem for each $n$. When there are an infinite number of terms in (4.2), an extra estimate is required to bound the tail of the sum.

Defining the projection $P_N = \sum_{n=1}^{N} |g_n\rangle \langle g_n|$, then we can write

$$
\frac{1}{i\gamma}(e^{i\gamma X} - I)C - XC = \sum_{n=1}^{N} \lambda_n \left( \frac{1}{i\gamma}(e^{i\gamma X} - I) - X \right) |f_n\rangle \langle g_n| - XC(I - P_N) + \frac{1}{i\gamma}(e^{i\gamma X} - I)C(I - P_N).
$$

Applying the triangle inequality to the above, we have that

$$
\| \frac{1}{i\gamma}(e^{i\gamma X} - I)C - XC \|_1 \leq \sum_{n=1}^{N} |\lambda_n| \| (\frac{1}{i\gamma}(e^{i\gamma X} - I) - X) |f_n\rangle \langle g_n| \|_1
$$

$$
+ \| XC(I - P_N) \|_1 + \| \frac{1}{i\gamma}(e^{i\gamma X} - I)C(I - P_N) \|_1.
$$

(4.3)

Our strategy will be to pick, for each $\epsilon > 0$ a number $N_\epsilon$ such that the last two terms on the right are bounded by $\epsilon$, for $N \geq N_\epsilon$ and arbitrary $\gamma$. Since the first term, with $N = N_\epsilon$ vanishes as $\gamma \to 0$ by the reasoning above, the claim will follow.

Since $XC \in \mathcal{B}_1(\mathcal{H})$, we have $\|XC(I - P_N)\|_1 \to 0$, as $N \to \infty$. We can use this to bound the third term on the right-hand side of (4.3). Note that for $A = \frac{1}{i\gamma}(e^{i\gamma X} - I)$ and $B = X$, we have $0 \leq |A|^2 \leq |B|^2$ by functional calculus. It follows that $Y^*|A|^2Y \leq Y^*|B|^2Y$ for any $Y \in \mathcal{B}_\infty(\mathcal{H})$, and we will use this case when $Y_N = C(1 - P_N)$. Since $\cdot \mapsto \sqrt{\cdot}$ is an operator monotone function, we have $\sqrt{Y_N^*|A|^2Y_N} \leq \sqrt{Y_N^*|B|^2Y_N}$. With this equality and two applications of the definition of the trace norm, we have

$$
\|AY_N\|_1 = \text{Tr} \sqrt{Y_N^*|A|^2Y_N} \leq \text{Tr} \sqrt{Y_N^*|B|^2Y_N} = \|BY_N\|_1.
$$

It follows that $\|AY_N\|_1 \to 0$ as $N \to \infty$ which completes the proof. \[\Box\]
We continue with the proof of Proposition 3.1.

Proof of Proposition 3.1:

Step 1: Assume the singular value decomposition for \( C \), as in (4.2). Then
\[
\kappa \mapsto \gamma_p(k) := \sum_{n \in \mathbb{N}} \lambda_n(e^{i\frac{\kappa}{2}X} f_n(k)(e^{-i\frac{\kappa}{2}X} g_n)(k) \tag{4.4}
\]
is a \( L^1 \)-function (by Cauchy-Schwartz, since it is summable series of products \( L^2 \)-functions), which depends continuously on \( p \), since \( p \mapsto e^{\frac{\kappa}{2}X} \) is a strongly continuous group on \( L^2(\mathbb{T}^d) \). It is straightforward to verify that \( [C]_p := \gamma_p \) satisfies our definition of the fiber decomposition (2.8). In other words,
\[
[C](k_1, k_2) := [C]_{k_2 - k_1}(\frac{k_1 + k_2}{2}) \tag{4.5}
\]
is a kernel for the operator \( C \).

Step 2: We show first that, under assumption (3.10), \([C]_p \) is actually in \( C_1 \). First, we show that, if both \( C \) and \( X \) \( X \) \( X \) are in \( B_1 \), then
\[
\frac{1}{2}[\{X_j, C\}] = \frac{\partial}{\partial p_j}[C]_p. \tag{4.6}
\]
By Step 1, it suffices to show that, in \( B_1 \),
\[
\frac{1}{i\lambda'(e^{i\lambda X} - 1)} C \rightarrow X \mathcal{C}, \tag{4.7}
\]
which is proven in Lemma 4.1. The continuity of \( \frac{\partial}{\partial p_j}[C]_p \) in \( p \), follows from (4.6) and the general argument in Step 1, with \( C \) replaced by \( \frac{1}{2}[C, X] \).

Step 3: The existence and continuity of the second derivative follows by repeating Step 2, since \([C, X]X \) and \( X[C, X] \) are in \( B_1 \).

The following lemma lays out the standard perturbation theory\(^{16}\) that we make use of.

**Lemma 4.2**: Consider a family of bounded operators \( L_p \) on a Banach space for \( p \in \mathbb{T}^d \). Assume that the \( \frac{\partial}{\partial p}_{|p=0} L_p \) are bounded and continuous as a functions of \( p \) in some neighborhood of \( 0 \in \mathbb{T}^d \). Finally, assume that \( \text{spec}(L_0) \), the spectrum of \( L_0 \), contains \( 0 \) as an isolated point, corresponding to a simple eigenvalue.

Then, for \( |p| \) small enough, the operator \( L_p \) has a unique eigenvalue \( D_p \) such that
\[
0 \leq D_p = (p, \text{tr}(T^{(1)} P_0)) + (p, \text{tr}(T^{(2)} P_0)p) - (p, \text{tr}(T^{(1)} S T^{(1)} P_0)p) + o(p^2), \tag{4.8}
\]
where \( T^{(1)} \) and \( T^{(2)} \) are, respectively, a vector of operators and a self-transpose matrix of operators (i.e., \( T^{(2)}_{i,j} = T^{(1)}_{j,i} \)), defined by the expansion
\[
L_p - L_0 = (p, T^{(1)} + (p, T^{(2)} p) + o(p^2),
\]
\( P_0 \) is the projection corresponding to the eigenvalue \( 0 \) of \( L_0 \), and \( S \) is the reduced resolvent of \( L_0 \) at the eigenvalue \( 0 \), i.e., the solution of
\[
S(0 - L_0) = (0 - L_0)S = 1 - P_0, \quad SP_0 = P_0 S = 0.
\]
Moreover, \( \text{spec}(L_p) \) \( \gamma \) \( \{D_p\} \) lies at a distance \( o(p^0) \) from \( \text{spec}(L_0) \) \( \{0\} \).\footnote{This proof is adapted from Reference 16.}

For our model the projection \( P_0 \) has the form \( P_0 = |\mathcal{P}| \mathcal{P} |_{1 \mathcal{T}^d} \), where \( \mathcal{P} \) is the stationary density for the Markov process of the zero fiber and \( 1 \mathcal{T}^d \) is the indicator function over \( \mathbb{T}^d \). The first-order perturbation term \( T^{(1)} \) is given in (2.17). The second-order perturbation \( T^{(2)} \) is given by (2.18) and it only depends on the map \( \Psi \), not on \( H \). With the assumption of (3.3) for \( W = UV \), so that the
second term on the right-hand side of (2.17) is zero, we have the following explicit expressions:

$$\text{tr}[T^{(1)} P_0] = i \int_{\mathbb{T}^d} dk \mathcal{P}(k)(\nabla H)(k),$$

(4.9)

$$\text{tr}[T^{(2)} P_0] = -\int_{\mathbb{T}^d} dk \int_{\mathbb{T}^d} dv(\theta) \mathcal{P}(k)\mathcal{M}_{\theta}^{(2)}(k),$$

(4.10)

$$\text{tr}[T^{(1)} ST^{(1)} P_0] = -\int_0^\infty dt \int_{\mathbb{T}^d} dk \zeta(k)(e^{iL_0 \zeta} \mathcal{P})(k),$$

(4.11)

with the function \( \zeta \) as defined in Proposition 2.4. We have used that on the range of \( S \), the integral \( \int_0^\infty dt e^{iL_0 t} \) is well-defined and equal to \( S \).

The expression (4.10) is non-positive since the matrices \( \nabla_1 \nabla_2 M_0(k, k) \) are non-negative, as explained in Sec. III A. Seeing that the expression (4.11) is non-positive is a little more tricky, but it helps to rewrite the right-hand side as the following:

$$\int_0^\infty dt \int_{\mathbb{T}^d} dk (e^{iL_0 \zeta})(k)\zeta(k)\mathcal{P}(k) = \int_0^\infty dt E_\mathcal{P}[\zeta(Y_t)\zeta(Y_0)].$$

The evolution has been shifted to operate on the observables which can then be rewritten in terms of the expectation \( E_\mathcal{P} \) of the Markov process \( Y_t \), started from the stationary distribution \( \mathcal{P} \). We show the non-negativity of this expression in the proof of Proposition 2.4.

Proof of Theorem 2.3: To prove convergence in distribution for \( \mu_t \), we show pointwise convergence of the characteristic functions.

$$\varphi_{\mu_t}(\gamma) := \int_{\mathbb{R}^d} d\mu_t(x) e^{i\gamma x} = \text{Tr}[\rho_t e^{i\gamma(x-x_0)}] = e^{-i\sqrt{\gamma} \mathcal{P}}\langle 1_{\mathbb{T}^d}, [e^{ix}(\rho)]^{(1)}_\mathcal{P} \rangle,$$

(4.12)

where the third equality makes use of the fiber decomposition of our dynamics. In particular, we used the relation (cf. the proof of Proposition 3.1)

$$[e^{i\frac{\gamma}{2}X} C e^{i\frac{\gamma}{2}X}]_p = [C]_{p+\gamma}, \quad C \in B_1.$$

(4.13)

For a fixed \( \gamma \) the limit involves only small fibers \( p \sim t^{-\frac{1}{2}} \), which suggests using a perturbation argument around the zero fiber. The continuity of the second derivatives of \( L_p \) follows from the continuity assumptions on the second derivatives of the functions \( M_0 \) and \( H \) in Assumption 2.1 and the forms (3.8) and (3.9). Lemma 4.2, for a small enough neighborhood of \( p = 0 \), say \( \mathcal{U} \subset \mathbb{T}^d \), we have that

$$[e^{iL_p} \rho]_p = e^{iL_p} [\rho]_p = P_p e^{iL_p} + (1 - P_p) e^{iL_p} [\rho]_p, \quad V_p := L_p - D_p P_p,$$

(4.14)

and \( \|e^{iL_p}\| = O(e^{-tb}) \) as \( t \not\to \infty \), for some \( b > 0 \) satisfying \( b - b_0 \to 0 \) as \( p \not\to 0 \). The norm refers to the operator norm on \( B(L^1(\mathbb{T}^d)) \) and \( b_0 \) is the gap of the operator \( A \) (Assumption 2.2).

We now show that, in \( L^1(\mathbb{T}^d) \),

$$e^{-i\sqrt{\gamma}\mathcal{P}}[e^{iL} \rho]_\mathcal{P} \xrightarrow{\gamma \not\to \infty} e^{-\frac{1}{2}i\langle p, \sigma \gamma \rangle} \mathcal{P}.$$}

(4.15)

This follows by combining (4.14), the relation

$$D_p = i(p,v) - \frac{1}{2}(p,\sigma p) + o(p^2),$$

(4.16)

(which follows from Lemma 4.2), and the fact that

$$P_p \xrightarrow{i \not\to \infty} P_0 = |\mathcal{P}|(1_{\mathbb{T}^d}), \quad [\rho]_p \xrightarrow{i \not\to \infty} [\rho]_0,$$

(4.17)

where the first convergence is in \( B(L^1(\mathbb{T}^d)) \) and the second in \( L^1(\mathbb{T}^d) \). The first claim of (4.17) follows again from Lemma 4.2, the second is a consequence of Proposition 3.1. We have shown
Using Bochner’s theorem we just need to check that

\[ \text{(4.15)} \]

and we obtain

\[ (4.15). \]

Since norm convergence implies weak convergence, in particular, \([e^{tL}(\rho)]_p\) \(\rightarrow\) integrated

against the indicator 1 \(\rightarrow\) converges to the desired value. Hence, \(\mu_t\) converges in distribution.

We now prove the convergence of the first and second moments (2.21). By (4.12), and the usual connection between moments and derivatives of the characteristic function, we have (whenever the right-hand side exists),

\[
\frac{1}{t} \sum_{x \in \mathbb{Z}^d} \rho_t(x, x)(x_i - t v_i)(x_j - t v_j)
\]

\[
= \frac{1}{t} \left( - \frac{\partial^2}{\partial p_i \partial p_j} \langle 1, e^{tL_p}[\rho_0]_p \rangle \bigg|_{p=0} + \frac{\partial}{\partial p_i} \langle 1, e^{tL_p}[\rho_0]_p \rangle \bigg|_{p=0} \text{d}p_j \right),
\]

\[ (4.18). \]

Note that since the operator \(L_p\) has two continuous derivatives, the operators \(P_p, V_p\) (defined above) and the eigenvalue \(D_p\) do also. In particular, \(\|\langle \frac{\partial^2}{\partial p_i \partial p_j} \rangle V_p \| \) is bounded for \(p \in \mathcal{U}\). One can see that,

\[
\sup_{p \in \mathcal{U}} \|\langle \frac{\partial^2}{\partial p_i \partial p_j} \rangle V_p \| = O(t^2 e^{-bt}), \quad t \not\to \infty.
\]

(and a similar bound for the first-order derivatives). By Lemma 3.1, the function \(p \mapsto [\rho]_p\) is \(C^2\) and we obtain

\[
\frac{\partial^2}{\partial p_i \partial p_j} \langle e^{tV_p}[\rho]_p \rangle \bigg|_{p=0} \not\to 0.
\]

Hence, in (4.18) we can replace \(e^{tL_p}\) with \(P_p e^{tD_p}\), for large \(t\), and we see the convergence to \(\sigma\) using the expansion (4.16).

\[ \square \]

Proof of Proposition 2.4:

Proof of Statement (1): In the proof of Theorem 2.3, we showed the pointwise convergence of the characteristic function \(\varphi_{\mu_t}\), i.e.,

\[
\varphi_{\mu_t}(\gamma) = \int_{\mathbb{R}^d} d\mu_t(x)e^{ix\gamma} \not\to e^{-t/2\gamma_\sigma}\gamma}
\]

(see, e.g., (4.12) and (4.15)). Suppose there were a \(\gamma \in \mathbb{R}^d\) such that \(\gamma_\sigma < 0\), then, for large enough \(t\), \(\varphi_{\mu_t}(\gamma) > 1\), which is impossible since \(\varphi_{\mu_t}\) is the characteristic function of a probability measure. This proves the non-negativity of the diffusion matrix \(\sigma\).

\[ \square \]

Proof of Statement (2): Now we consider the non-negativity of the matrix \(\alpha\). We show that \(\alpha\) is a non-negative matrix by showing that the expression \((w, \alpha w)\) for \(w \in \mathbb{R}^d\) is always non-negative. Using an unsymmetrized form for \(\alpha\), we can rewrite our evaluation as

\[
(w, \alpha w) = \int_0^\infty dt \mathbb{E}_t[f(Y_t)f(Y_0)],
\]

(4.20)

where \(f(k) := (w, \zeta(k))\) is real valued. Let the stationary Markov process \(Y_t\) be extended to all negative values of \(t\). Then \(G(t) = \mathbb{E}_t[f(Y_t)f(Y_0)]\) is an even function and so (4.20) is twice the value of the Fourier transform \(\hat{G}(z)\) of \(G(t)\) at \(z = 0\). We will show that \(\hat{G}(z)\) is non-negative valued. Using Bochner’s theorem we just need to check that \(G(t - s)\) defines a positive operator on \(L^2(\mathbb{R})\).

Let \(\eta \in L^2(\mathbb{R})\), then

\[
\int_{\mathbb{R}^2} dt ds \bar{\eta}(t)G(t - s)\eta(s) = \mathbb{E}_t[\int_{\mathbb{R}} dt \eta(t)f(Y_t)^2],
\]

where we have used the stationarity property \(\mathbb{E}_t[f(Y_{t-s})f(Y_0)] = \mathbb{E}_t[f(Y_t)f(X_s)]\).

\[ \square \]
Proof of Statement (3): The strict positivity of $\alpha$ is established as follows. Let $w \in \mathbb{R}^d$. By the assumption that the velocity fluctuates, $g(k) := (w, \xi(k)) \in \mathcal{H}$ satisfies

$$g \neq 0, \quad (1_{T^d}, g)_{\mathcal{P}} = 0,$$

where $1_{T^d} \in \mathcal{H}$, the identity function on $T^d$, is the 0-eigenvector of $A_{\mathcal{P}}$. Note further that

$$(w, \alpha w) = \langle g, (A_{\mathcal{P}})^{-1} g \rangle_{\mathcal{P}},$$

(4.21)

since one can easily check that $h := (A_{\mathcal{P}})^{-1} g \in \mathcal{H}$ by using the fact that $A_{\mathcal{P}}$ is bounded and $A$ has a gap. Assume that $\alpha$ is not strictly positive. Then, there is a $w \in \mathbb{R}$ such that (recall that $g$ and hence $h$ depend on $w$)

$$\langle g, (A_{\mathcal{P}})^{-1} g \rangle_{\mathcal{P}} = \langle A_{\mathcal{P}} h, h \rangle_{\mathcal{P}} = 0.$$

In particular, this implies that

$$\langle \text{Re}(A_{\mathcal{P}}) h, h \rangle_{\mathcal{P}} = 0.$$  

(4.22)

Since $-\text{Re}(A_{\mathcal{P}})$ is a positive operator (as follows from the fact that $A_{\mathcal{P}}$ is a Markov generator), we get $\text{Re}(A_{\mathcal{P}}) h = 0$. By the sectoriality assumption, it follows also that $\text{Im}(A_{\mathcal{P}}) h = 0$ and hence $A_{\mathcal{P}} h = 0$. Indeed, for $\lambda > 0$ and $v \in \mathcal{H}$,

$$\langle (h + \lambda v), \text{Im}(A_{\mathcal{P}})(h + \lambda v) \rangle_{\mathcal{P}} \leq \gamma \| (h + \lambda v) \|, \quad \text{Re}(A_{\mathcal{P}})(h + \lambda v) \rangle_{\mathcal{P}} = \gamma \lambda^2 \langle v, \text{Re}(A_{\mathcal{P}}) v \rangle_{\mathcal{P}}$$

and hence the $O(\lambda)$-term has to vanish on the left-hand side for all $v$. Since the zero-eigenvector of $A_{\mathcal{P}}$ is unique by assumption, we obtain $h = c 1_{T^d}, c \in \mathbb{R}$. This leads to a contradiction with the fact that $h = (A_{\mathcal{P}})^{-1} g$ and $(1_{T^d}, \mathcal{P}) = 0$. Hence $\alpha$ is strictly positive. □

Note that (4.20) is related to the familiar central limit theorem for Markov processes

$$\lim_{T \to \infty} \frac{1}{\sqrt{T}} \int_0^T dt f(Y_t) \to \mathcal{N}(0, \sigma),$$

where the convergence is in distribution, $\sigma = \int_0^\infty dt \mathbb{E}_\mathcal{P}[f(Y_t) f(Y_0)]$, and the function $f$ satisfies $f_{\mathcal{T}^d}, dk f(k) \mathcal{P}(k) dk = 0$.

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