ENTROPY—A FABLE
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A scientist who was always very curious about the relation between entropy and information but was afraid to ask, dies and is ushered into a very large room where there sits behind a desk the angel of knowledge. On the desk there is a shiny sphere with some special aura surrounding it. The eye of the scientist (S) is attracted to the sphere and he asks what is in it. The angel (A) answers.

A: It’s a ball filled with one heavenly mole \(2^{31}\) of very small hard balls: they occupy a volume fraction \(2^{-18}\) of the outer ball. These particles move according to Hamiltonian dynamics with perfectly elastic collisions and are perfectly shielded from all external influences. I started them in a particular microscopic state \(X\), which I picked at “random” from a given region \(\Omega\) of the phase space, a very very long time ago; the phase points in \(\Omega\) correspond to all the particles being in a ball with the same origin and one tenth the radius of the big ball, with a total energy between \(E \pm \Delta E\), \(\Delta E = 2^{-18} E\). I have watched them evolve since then out of curiosity for how such a classical system would behave.

This is great, thinks S. Here is my chance to find out about entropy and information but I better check first if there is any trick here.

S: How interesting. I assume that your computers, I mean brains, are big enough and accurate enough so that you know the exact position and velocity of each particle at every instant of time.

A: Yes, of course.

S: But then you also know in advance how it will evolve. Why do you bother with this
experiment?

A: You see, while I have essentially unlimited memory and therefore have perfect knowledge of the past and present state of the system, I do not have any special computational abilities and therefore am unable to predict the future any better than mortals are. It is therefore fun to watch it and show it off to an interested visitor.

S: O.K., so let's discuss it. According to my understanding the initial time evolution should have been described very accurately by the Boltzmann equation (BE) and since then just equilibrium. The fluctuations are quite well described by the linearized BE with Gaussian noise.

A: Absolutely. That is exactly how it looked on the mesoscopic or kinetic scale.

S: Fine, but let me ask you: are you really sure that your system is truly isolated?

A: Yes. Just to be sure I occasionally reverse all the velocities and let the system evolve for a few years and then reverse the velocities again. The particles all retrace their trajectories perfectly.

S: Have you ever seen them all go back spontaneously into the little ball where you started them from?

A: Oh, you mean a Poincaré recurrence. Actually no, even I am not quite old enough to have witnessed such an event, but I am waiting. It would be really nice to see that but you know, I will be replaced by another angel much before then.

S: And just before that would happen how will the entropy of the system behave?

A: Why do you ask? It will decrease just as it increased after I started them and will increase again after that passes. In fact, as I told you, I can myself organize a perfect
recurrence by just reversing all velocities and all the particles get back in their initial small region of the ball. Then, the entropy has dropped to its initial small value.

S: I am puzzled. To help me clarify things, let me begin my asking you what is the present entropy of this system?

A: Looking at it at this instant it seems to be pretty uniform and have a Maxwellian distribution of velocities, so the macrostate is the equilibrium one. Hence if you want the entropy, I would just take the log of the phase space volume appropriate to its energy, and get the value given in the textbooks.

S: This is what has been puzzling me for a long time. According to the textbook this is the entropy appropriate for ignorant mortals who have to use some phase space probability density \( \mu \). Given such probability, they compute the entropy via \(-\int \mu \log \mu dX\); this formula, first introduced by Gibbs and later extended by Shannon and others, has been almost synonymous with entropy in our world. You on the other hand know the microscopic state, so your probability density is in effect a delta function, why shouldn't then your entropy be \(-\infty\)? Or, if you use some discretization to take into account quantum mechanics or whatever, then it should be zero. Why should the volume of the energy surface be relevant for you?

A: Oh, but I was not talking for anybody in particular. Of course I do know, to any precision I care, the exact locations and velocities of the particles. I thought you were asking me about the present macrostate corresponding to the energy and density profile of the system and how many microstates, i.e., phase space volume, fit in it. Obviously, no matter what we really know, this quantity is unambiguously defined (up to relatively
negligible terms) which depend on the precision with which you define your macrostate, and it gives rise to the standard entropy.

S: What do you mean by standard entropy? Are there different entropies?

A: Well of course. Not only are there different types of entropy that are useful for different types of problems but within many of these you have still differences depending on what relevant macrovariables you take.

S: You probably mean that you have something like the measure-theoretic entropy, the topological entropy, the von Neumann entropy, the Gibbs entropy, the H-functional, the Clausius thermodynamic entropy, the Boltzmann entropy etc. I have always felt that it is unfortunate to have so many with the same name but what is your point?

A: These I call the different types. Here we are speaking in a statistical mechanical context and I was referring to the Boltzmann entropy and also to the thermodynamic equilibrium entropy. You must know that these are only defined once you specify the macrovariables. So if you give me another set of macrovariables, you get a different Boltzmann entropy.

S: But is this choice completely arbitrary? In the textbooks I have always encountered the same things.

A: Yes and no, it depends for what purposes. If you want to use it as a predictive tool for the time evolution as in the second law then it is closely linked with the type and level of description you are using. Of course you are free but certain choices make much more sense. In fact, for thermodynamics I would even say you have a rather limited choice.

S: Then what determines a good choice.

A: What you are really asking is what makes a function on the microstate a good macrovari-
able. One thing that is crucial is to have a function that is additive. The reason is that then most of phase space, say corresponding to a given energy, is occupied by one very large region where these macrovariables are constant. It also helps to use quantities which are conserved locally. This essentially selects the energy, momentum and particle number.

S: How do you measure these volumes in phase space?
A: You count. It is a bit tricky but you can for all practical purposes also take some nice density over which you integrate.

S: Then what would be an example of having different entropies in your sense?
A: There are many but here is a trivial one. Suppose that I have here red and green particles and that initially, all red particles were to the right of my box and all green particles were to the left. Same pressures, same temperatures. Now I let them go and they mix. I get something brown. Did the entropy increase?

S: This I know. Of course the entropy increases. I can even compute by how much.
A: Aha, but now suppose that you were colorblind, you could not see the difference between green and red and you would see something pretty uniformly grey from beginning to end.

S: And the particles can be assumed to be individually taking each exactly the same paths as before?
A: Yes. The microscopics is identical.

S: Strange. Now I would say there is no increase in entropy. Is that not related to the Gibbs paradox?
A: Indeed, as is its solution. You see, there are many entropies. In one case you add the extra macrovariable giving you the red density-profile and if you do it colorblindly, you
just inspect the overall density profile.

S: There is one thing I am now confused about. If there are these different entropies corresponding to what you and I decide to include as macrovariables, how can the measurement of entropy be objective, yielding the same result for all of us?

A: That is a good question. The solution is that you should think about what is relevant and what is not for the problem you are considering. The possibility of having different entropies is not at all harmful as long as you understand that it corresponds to different situations, e.g. are you interested in properties of the system which depend on the color of the particles or not? Of course in the former case you better have someone or some instrument which is color sensitive.

S: Do you mean something like contextual?

A: Something like that but don’t make it too complicated. Just imagine a colorblind man trying to sort out green from red particles. As for contextuality, you also have it in quantum mechanics. Maybe it is to this that you refer.

S: That reminds me. As you know everything, is there still something like indistinguishability for you?

A: No, I can distinguish all particles.

S: That is what I thought but then how do you get quantum statistics and why do you divide by this $N!$ in front of the Liouville measure?

A: Now it is me not understanding your question.

S: I have seen many textbooks arguing for extensivity of the entropy via quantum mechanics and I have learnt that quantum particles follow a Bose-Einstein or Fermi-Dirac statistics
for their occupation numbers. But they must be indistinguishable.

A: Did that not sound a bit strange to you?

S: No no, it looked pretty convincing.

A: But they did not know about me, it seems. The problem of extensivity of entropy is unchanged in going from classical to quantum physics.

S: Are you saying that this indistinguishability is similar to your example of colorblindless where you can exchange red for green?

A: It is. You see, also classical particles with the same charge, mass and what have you, don’t come equipped with distinguishing labels. They are in fact only distinguished by their dynamic properties. That is the history of their momenta and positions. There is no reason therefore to be mystical about this $N!$; it emerges naturally in classical mechanics just like in quantum mechanics.

S: Perhaps so but, at any rate, quantum mechanics is much more complicated. Since there is no obvious quantum phase space of microstates there is also no obvious phase space volume so tell me, how should one define the quantum Boltzmann entropy?

A: Boltzmann speaks about the connection between the microscopic and macroscopic thermal properties of physical systems. This can be divided into two parts. The first part gives a microscopic formula for computing the entropy $S_B(M)$ of a macroscopic system specified to be in a macrostate $M$, e.g. one described by the hydrodynamic variables entering the Navier-Stokes equation. This goes over without any deep conceptual problems to quantum mechanics. Given $M$ you can still use Boltzmann’s formula $S_B(M) = k \log |\Gamma(M)|$ where $|\Gamma(M)|$ is now the dimension of the linear subset of the Hilbert space of the system corre-
sponding to $M$. This makes sense because the variables defining the macrostate, e.g. those specifying (to some appropriate accuracy) the particle number, the momentum and the energy in each macroscopically small but microscopically large region of the volume occupied by the system, can be chosen to essentially commute and so they form an orthogonal decomposition of the Hilbert space.

S: That’s great. I actually remember reading about that in von Neumann’s book but had forgotten about it since all the textbooks and papers always use for the entropy the formula $-kTr(\hat{\mu}\log\hat{\mu})$ where $\hat{\mu}$ is the density matrix but that is zero if you are in the pure state $\Psi$.

A: Yes. This quantum version of the Gibbs entropy, also called the von Neumann entropy, has produced even more confusion than its classical counterpart. But aside from this unnecessary confusion you, I mean your colleagues still down there, do have a real problem when it comes to the second part of Boltzmann’s micro-macro connection. Given a point $X$ in the phase space we always have a well defined $M$: call it $M(X)$. This defines the Boltzmann entropy $S_B(X) = S_B(M(X))$ for a single macroscopic system like the hard spheres inside this box. As you know, this is important for giving a clear microscopic interpretation to the second law. There is no such relationship between $\Psi$ and $M$ for quantum systems: the wavefunction of the system does not necessarily specify a unique macrostate. Think of Schrödinger’s cat.

S: That is exactly what I was just thinking about. How can I understand this?

A: Well, to tell you the truth I don’t really understand it either. I think that there are other ways to think about quantum mechanics but I am not really the right angel to explain
them. Why don’t you see my colleague Angel Psi.

S: Good. I want to see Angel Psi. I have many questions for him.

A: I wish you good luck.

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[8] J.C. Maxwell, Scientific Letters and Papers, Ed. P. M Harman, (Cambridge University Press, 1990-95). In Volume II, pages 331-332, Maxwell’s demon is introduced. This demon, or, as Maxwell himself wished to call it, Maxwell’s valve or finite being is not unlike the angel in the fable. Maxwell emphasizes that his purpose in invoking this thought experiment was to explain that the second law of thermodynamics has only a statistical certainty. See also Tait’s Thermodynamics, Nature 17, 257 (1878).

